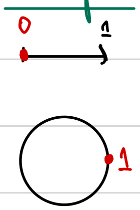
Math 564: Real analysis and measure theory
Lecture 12

Pushforward measures.

Def. Let $(X, \mathcal{I})$ and $(Y, \mathcal{J})$ be measurable spaces and let μ be a measure on $\mathcal{I}$. Let $f : X \to Y$ be an $(\mathcal{I}, \mathcal{J})$-measurable function. Then the *f-pushforward of μ* is the measure $f_* \mu$ (also μf^{-1}) on $\mathcal{J}$ defined by: for $J \in \mathcal{J}$,
$$f_* \mu (J) := \mu(f^{-1}(J)).$$

Examples. (a) Let $S^1 \subseteq \mathbb{C}$ denote the unit circle, which is usually considered a group under complex multiplication. This is identified with the group $(\mathbb{R}/\mathbb{Z}, +)$ as follows: $\mathbb{R}/\mathbb{Z} \cong [0,1)$ since $[0,1)$ is a transversal for the coset eq. rel. of $\mathbb{Z} \leq \mathbb{R}$. Define $f : [0,1) \to S^1$ by $x \mapsto e^{2\pi i x}$, which is a group isomorphism $(\mathbb{R}/\mathbb{Z}, +) \xrightarrow{\sim} (S^1, \cdot)$. Then $f_* \lambda$ is a Borel measure on S^1.



(b) Let $\mathbb{R}_{>0}$ be the group of positive reals under $\cdot$. Consider $f : \mathbb{R} \to \mathbb{R}_{>0}$ by $x \mapsto e^x$ and take the pushforward measure $f_* \lambda$. In particular, for $(a, b) \subseteq \mathbb{R}_{>0}$,
$$f_* \lambda((a,b)) = \lambda((\log a, \log b)) = \log b - \log a.$$

Def. Let $(X, \mathcal{I}, \mu)$ be a measure space.

- For an $(\mathcal{I}, \mathcal{I})$-measurable function $f : X \to X$, we say that μ is *f-invariant* or that f *preserves* μ if $f_* \mu = \mu$, i.e. $\mu(B) = \mu(f^{-1}(B))$ for all $B \in \mathcal{I}$.
- For a group action $\Gamma \curvearrowright X$ such that each group element acts as an $(\mathcal{I}, \mathcal{I})$-meas. function, we say that μ is *Γ-invariant* or Γ *preserves* μ if for each $\gamma \in \Gamma$, $\gamma_* \mu = \mu$.

Example. Let $s : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be the *left-shift* map, i.e. $(x_n)_{n \in \mathbb{N}} \mapsto (x_{n+1})_{n \in \mathbb{N}}$. Any Bernoulli

measure $\nu^{\mathbb{N}}$ is shift-invariant: indeed, it suffices to check on cylinders $[w]$:
$$s^{-1}([w]) = \bigsqcup_{a\in A} [aw], \quad \text{so} \quad \nu^{\mathbb{N}}(s^{-1}([w])) = \sum_{a\in A} \nu^{\mathbb{N}}([aw]) = \sum_{a\in A} \nu(a)\cdot \nu^{\mathbb{N}}([w]) = \nu^{\mathbb{N}}([w]).$$

A topological group is a group G equipped with topology making multiplication and inverse continuous. A Borel measure μ on G is called left-invariant (resp. right-invariant) if it is invariant under the left-translation (resp. right-translation) action $G \curvearrowright G$, i.e. $\mu(g\cdot B) = \mu(B)$ (resp. $\mu(B\cdot g) = \mu(B)$).

Theorem (Haar). Every locally compact (Hausdorff) group admits a unique (up to scaling) locally finite left-invariant Borel measure (also a right-invariant Borel measure).
($\Updownarrow$ finite on compact sets) This measure is called a left Haar measure (resp. right Haar measure).

Examples. (a) For $(\mathbb{R}^d, +)$, Lebesgue measure is a Haar measure.

(b) For $(\mathbb{R}_{>0}, \cdot)$, the pushforward of Lebesgue by $x \mapsto e^x : \mathbb{R} \to \mathbb{R}_{>0}$ is a Haar measure because this function is a topological group isom. and Lebesgue is Haar for $(\mathbb{R}, +)$.

(c) For $(S^1, \cdot)$, the pushforward of Lebesgue by $x \mapsto e^{2\pi i x} : [0,1) \to S^1$ is a Haar measure because this function is a topological group isomorphism and Lebesgue measure is a Haar measure on $\mathbb{R}/\mathbb{Z} \simeq [0,1)$. Note that this measure is a probability measure, hence this is the unique probability Haar measure.

(d) Consider the group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} \cong 2^{\mathbb{N}}$ as a compact group with the same topol. as $2^{\mathbb{N}}$, under coordinatewise addition mod 2. The Bernoulli$(\frac{1}{2})$ measure is the Haar probability measure on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$. Incidentally, this is also the pushforward of Lebesgue on $[0,1) \cong \mathbb{R}/\mathbb{Z}$ by $f : [0,1) \to 2^{\mathbb{N}} \cong (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ by $x \mapsto$ binary repr. of x.

Nonexample. The group $GL_n(\mathbb{R})$ of all invertible real matrices under multiplication is locally compact when viewed as a subset of $\mathbb{R}^{n^2}$: indeed, $M \in GL_n(\mathbb{R}) \iff \det(M) \neq 0$, and the latter is an open condition, so $GL_n(\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$. Furthermore, its complement, the $\det = 0$ set, is "lower dimensional" closed set, and one can show that it's null, so $GL_n(\mathbb{R})$ is a Lebesgue conull open subset of $\mathbb{R}^{n^2}$. However, Lebesgue measure on $GL_n(\mathbb{R})$ is not a Haar measure (neither left nor right) because for example multiplication by $\begin{pmatrix} 2^{\frac{1}{n}} & & 0 \\ & 2^{\frac{1}{n}} & \\ 0 & & \ddots \, 2^{\frac{1}{n}} \end{pmatrix}$ scales the Lebesgue measure by 2. The Haar measure on $GL_n(\mathbb{R})$ is defined using the Jacobian, i.e. integral with $\frac{1}{\det}$ in it.

We showed that translation actions $\mathbb{Q} \curvearrowright \mathbb{R}$ and $\bigoplus_{n\in\mathbb{N}}(\mathbb{Z}/2\mathbb{Z}) \curvearrowright \prod_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ are ergodic, and one can also similarly show that for an irrational $\lambda \in \mathbb{R}\setminus\mathbb{Q}$, the rotation by $2\pi\lambda$ on S^1 is ergodic, which is the same as the translation action of the (dense) subgroup $\langle e^{2\pi i\lambda}\rangle \leq S^1$ on S^1. The following shows that this is a general phenomenon:

Theorem. Let G be a locally compact (Hausdorff) group and $\Gamma \leq G$ be a dense subgroup. Then the (left) translation action $\Gamma \curvearrowright G$ is ergodic with respect to any Haar-measure.

Borel/measure isomorphism theorems.

The following is one of the basic theorems in descriptive set theory, which is used by mathematicians (e.g. ergodic theorists and probability theorists) all the time without mention.

Borel isomorphism theorem. Any two unctbl Polish spaces are Borel isomorphic, i.e. $\exists\, f : X \to Y$ a bijection s.t. f and f^{-1} are Borel.

Proof-sketch. For an unctbl Polish space X, it's enough to show that X is Borel isomorphic to $2^{\mathbb{N}}$. By the Borel version of the Cantor–Schröder–Bernstein theorem, it is enough to show that $\exists$ Borel $2^{\mathbb{N}} \hookrightarrow X$ and $X \hookrightarrow 2^{\mathbb{N}}$. The first is called the Cantor-Bendixson theorem: for each uncbl Polish X, there is a continuous embedding $2^{\mathbb{N}} \hookrightarrow X$.

Lemma (binary representation). Any 2^{nd} ctbl metric space X admits a Borel

$$b : X \hookrightarrow 2^{\mathbb{N}}$$

which we call a binary representation map.

Proof. Let (U_n) be a ctbl basis for X, so it separates points. Then define $b: X \to 2^{\mathbb{N}}$ by $x \mapsto (\mathbb{1}_{U_n}(x))_{n\in\mathbb{N}}$. To check that b is Borel, it is enough to observe that $b^{-1}(V_n) = U_n$ is Borel, where $V_n := \{x \in 2^{\mathbb{N}} : x(n) = 1\}$, because $\{V_n\}$ generates $\mathcal{B}(2^{\mathbb{N}})$ (indeed, each cylinder in $2^{\mathbb{N}}$ is a finite intersection of these V_n and their complements). $\square$

This finishes the sketch of Borel isom. theorem. $\blacksquare$

Def. A measurable space $(X, \mathcal{I})$ is called a standard Borel space if there is a Polish metric on X such that $\mathcal{I} = \mathcal{B}(X)$. In other words, X was a Polish space, but we forgot its topology and only kept the Borel σ-algebra.

Thus the Borel isom. theorem says that there is only one (up to isomorphism) standard Borel space.

Def. Let $(X, \mathcal{I}, \mu)$ and $(Y, \mathcal{J}, \nu)$ be measure spaces. A function $f: X \to Y$ is called a measure isomorphism if there are conull sets $X' \subseteq X$ and $Y' \subseteq Y$ such that $(f|_{X'}) : X' \to Y'$ is a bijection such that $(f|_{X'})$ is $(\mathcal{I}, \mathcal{J})$-measurable and $(f|_{X'})^{-1}$ is $(\mathcal{J}, \mathcal{I})$-measurable and $f_* \mu = \nu$.

Measure isomorphism theorem. Every atomless Borel probability measure μ on a Polish space X is isomorphic to $([0,1], \lambda)$. In fact, there is a Borel isomorphism $f: X \to [0,1]$ with $f_* \mu = \lambda$.